Multisoliton propagation in a linear granular chain

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When m grains hit a linear Hertzian chain of identical grains, m solitons are generated. We studied the multisoliton propagation using a particle dynamic simulation. The speed of solitons depends not only on the number of colliding grains but also on the sequence of generation. We found the hierarchy and evolution of the solitons as well as the generation of secondary solitons. We also found the oscillation and beats in the kinetic energy of the chain, which come from the discreteness of the medium in comparison with the spatial spreading of the soliton.

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I. INTRODUCTION

One of the simplest granular systems might be the one-dimensional chain of monosized elastic spheres. But the system is already complex enough due to the interactions of the adjacent spheres. The interaction of two contacting spheres is nonlinear to compression. Even if the spheres are made of perfectly elastic material, the contact geometry of the spheres generates the nonlinearity. This kind of contact is called a Hertzian contact [1] and a one-dimensional array of Hertzian contacts a Hertzian chain.

When a sphere collides with the chain, there arises a local compression which propagates along the chain. There are two extreme cases. One is a linear approximation regime where the amplitude of the compression wave is much smaller than the static compression already present in the chain. The wave in this case can be approximated to a normal sound wave. The other case is a nonlinear limit where the amplitude is larger than the static compression. One example of this case is that when all the spheres are just touching each other, i.e., the static compression is zero. In this situation, all perturbations, no matter how small, will generate nonlinear wave. Nesterenko showed that the nonlinear compression wave is a soliton [2] as was verified by experiment [3].

It is well known that when one sphere collides with the linear chain at one end, a sphere at the opposite side is ejected from the chain. If two spheres collide, then two spheres will be ejected. How does the impact get transmitted through the chain? Single-soliton propagation in the Hertzian chain was studied extensively [4]. On the other hand, the multisoliton problem, whereby two or more solitons simultaneously propagate along the chain, has not been studied much [5,6]. In this work, we figure out what will occur when two or more spheres are colliding, and analyze the properties of the compressive solitons generated.

II. MODEL SYSTEM AND ANALYTIC RESULTS

The elastic force between two contacting objects can be expressed as $F=P\delta^n$, where δ is the overlap of contacts. If n is 1, we get the normal Hook's law of a linear spring and P is a spring constant. When n is equal to 3/2, it becomes the Hertzian contact force of spherical objects [1] and its P value is

$$P=\frac{2Y}{3(1-\sigma^2)}\sqrt{\frac{R}{2}}, \tag{1}$$

where Y is Young's modulus, σ is Poisson's ratio, and R is the radius of the spheres.

Let us consider a linear chain of spheres which are touching each other slightly. Each sphere will move according to the following the set of equations of motion:

$$mu_i''=P[u_{i-1}-u_i]^n-P[u_i-u_{i+1}]^n, \quad i=2,3,\ldots,N-1. \tag{2}$$

In the above, m is the mass of a sphere and u_i is the displacement of the center of the ith sphere from its initial equilibrium position. The bracket takes the argument value if it is positive. Otherwise it takes a value of zero. The first and second terms on the right-hand side of Eq. (2) will vanish for $i=1$ and N, respectively.

The velocity of the compression wave generated by a collision is given by [7]

$$c=C_0(n)(P/m)^{1/2}A^{(n-1)/2}, \tag{3}$$

where A is the amplitude of the compression wave and C_0 is a constant which is a function of n. When the force exponent n is equal to 1, which is the normal linear wave case governed by Hook's law, the propagation velocity c does not depend on the wave amplitude. If we express the amplitude A in terms of the impact velocity v_0 of the sphere using the energy conservation equation,

$$\frac{1}{2}mv_0^2=\frac{PA^{n+1}}{n+1}, \tag{4}$$

we get the impact velocity dependence as

$$c=C_0'(n)(P/m)^{1/(n+1)}v_0^{(n-1)/(n+1)}, \tag{5}$$

where

$$C_0'(n)=C_0(n)\left(\frac{n+1}{2}\right)^{(n-1)/2(n+1)}. \tag{6}$$

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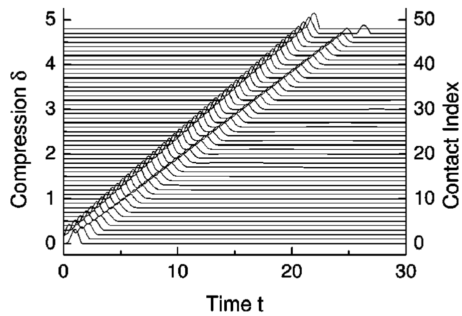

FIG. 1. Compression wave propagation as a function of time. The contact index starts from 0 to 48, with 50 spheres. See the text for the simulation parameters.

For the case of Hertzian contacts ($n=\frac{3}{2}$), the velocity of the compression wave becomes

$$c=C_0(\tfrac{3}{2})(P/m)^{1/2}A^{1/4}=C_0'(\tfrac{3}{2})(P/m)^{2/5}v_0^{1/5}. \quad (7)$$

In this study, we solve the equations of motion by direct numerical simulation. We applied the fourth-order Gear predictor-corrector algorithm [8] to a 50-sphere chain. When we needed more spheres for the velocity to settle down, we increased the grain number to 200. The simulation parameters are set to $P=10$, $m=1$, and $2R=1$. The infinitesimal integration time is set to 10^{-4}.

III. RESULTS

A. Velocity of solitons

When a sphere collides with the Hertzian chain, we found that the generated solitary wave has the same power exponents with respect to the impact velocity and the material constant as the continuum theory predicted. When two spheres collide, two solitons are generated with some time delay between them. Figure 1 shows the propagation of the two solitons as a function of time and space. Its simulation parameters are $P=10$, $v_{01}=1$, $v_{02}=1$, and $\Delta x_{12}(t=0)=0$, where $\Delta x_{12}(t=0)=0$ means that the two incoming spheres are just touching each other slightly.

The occurrence of the two solitons can be explained in the following manner. After the first sphere starts to collide with the chain at $t=0$, it is slowed down because of compression. The second sphere collides with the first one which is slowing down. This second impact generates a second compression of the sphere. These two compressions propagate and develop two solitary waves.

These two solitons have the same power dependence on the impact velocity and the material constants as in the case of one colliding sphere. But the velocities are different for the one and two colliding sphere cases even though we keep the impact velocity the same. The difference comes from the fact that the coefficient $C_0'(\frac{3}{2})$ in Eq. (7) depends on the number of colliding spheres and the sequential occurrence of the collisions. Figure 2 shows the hierarchical form of the coefficients.

If there is inhomogeneity such as local compression or mass difference, there will be an effective local potential

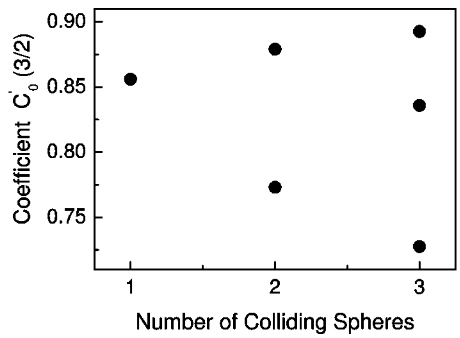

FIG. 2. Plot of the coefficient $C_0'(3/2)$ in Eq. (7) as a function of the number of colliding spheres. At one particular colliding sphere number, the early generated solitons have larger coefficient values.

barrier in the chain. When another compression wave passes through the barrier, there will be a scattering and variation of velocity [4]. When multiple solitons propagate, one particular soliton experiences the compression or the dilation (the noncontacting gap between the consecutive grains) of the chain which was made by its neighbors. We think that this scattering of overlapped potentials may cause the hierarchy of the coefficient.

It is known that the boundary effect makes the compressional wave achieve its asymptotic velocity after it passes five or six grains from the boundary [9]. We can see this transient period from the trace which is labeled as 11 in Fig. 3. On the contrary, multisolitons have a much longer transient period. While the soliton generated earlier becomes faster, the latter one becomes slower. This phenomenon is similar to the interaction of two solitons of Korteweg-de Vries type [5].

B. Generation of secondary solitons

Figure 4(a) shows the solitons generated by the collision of two slightly touching balls hitting the chain. Its magnified picture, Fig. 4(b), shows that the secondary soliton is generated at about the eighth contact from the impact point, and it is detached from the two primary solitons at around the 22nd contact. We observed that this is the same point where the

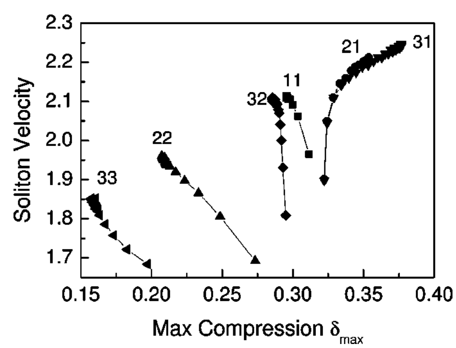

FIG. 3. Traces of maximum compression and its velocity at each contact point. The first value in the two-digit index represents the number of colliding spheres and the second one represents the sequence of generation. For example, the index 32 represents the second generated soliton from the collision of three spheres.

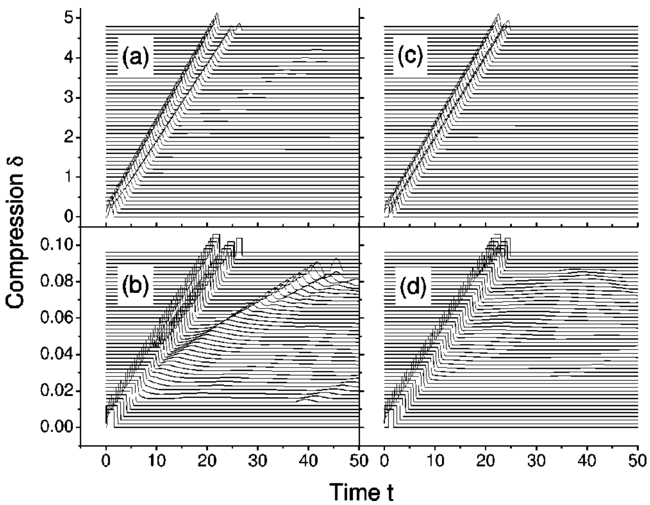

FIG. 4. (a) The propagation of two compression waves after two spheres collide with the chain with initial gap $\Delta x_{12}(t=0)=0$. (b) Magnified picture of (a). (c) Except for $\Delta x_{12}(t=0)=0.3$, the other conditions are the same as (a). (d) Magnified picture of (c).

velocity of the second soliton starts to decrease. Figure 4(c) has the same simulation conditions as (a) except for the initial gap of the two colliding grains, which is now $\Delta x_{12}(t=0)=0.3$. With finite Δx, the second impact is made with an additional time delay after the first one. Eventually, this additional time delay makes the two solitons have larger initial spatial distance as compared with the no-gap case. Its magnified picture (d) with resolution the same as (b) shows that there are much smaller but still noticeable secondary waves.

It was reported that when two oppositely directed identical solitons collide in the center of the chain, there appear secondary solitons which are weaker in magnitude and much slower than the primary ones [6]. Our results show that the generation of this secondary soliton is a general phenomenon which does not require two oppositely directed identical solitons. Only a slight touching of two propagating solitons is sufficient. In Fig. 5, the velocities of the two solitons approach that of a single soliton as $\Delta x_{12}(t=0)$ increases. But, the distance required for the solitons to reach their final velocity becomes longer. This implies that the repulsive interaction is decreased as the separation of the two solitons increases.

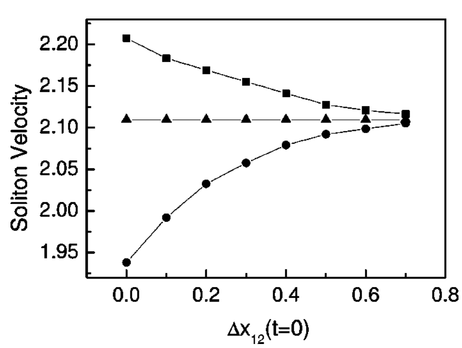

FIG. 5. As the initial gap $\Delta x_{12}(t=0)$ increases, all the velocities of solitons approach the value for the one-ball collision case. The upper one is the first generated soliton, and the lower one is the second generated one.

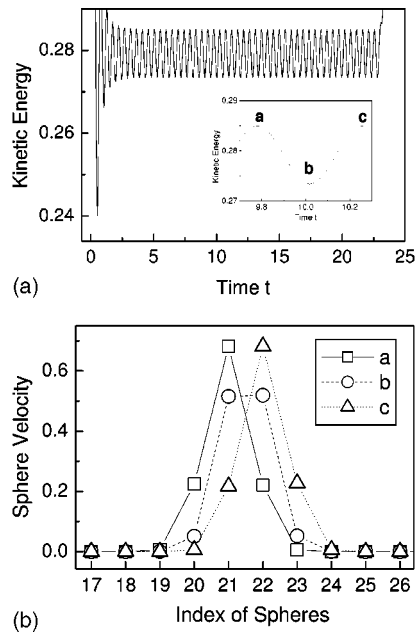

FIG. 6. (a) Kinetic energy of the chain along which a soliton propagates. The inset is an expanded figure for one full cycle. (b) Velocities of each of the spheres where the soliton passes at three different times corresponding to *a*, *b*, and *c* in (a).

C. Beats in energy

We checked the energy and momentum conservation in our numerical study. The total energy and momentum are conserved as expected. We find, however, oscillations in both the kinetic and potential energies. Figure 6(a) shows the kinetic energy oscillation during the propagation of a soliton along the chain. As previously mentioned, the boundary effect which appears as a transient part in the oscillation quickly fades out. The inset of the figure shows one full period of steady oscillation. Figure 6(b) shows the velocity profiles of the spheres at each time corresponding to *a*, *b*, and *c* in Fig. 6(a). Two consecutive spheres indexed as 21 and 22 have most of the positive velocity throughout the chain. The constraint of momentum conservation, $\Sigma p_i=\text{const}$, makes the kinetic energy have its minimum value when the velocities are evenly distributed. In our example of Fig. 6(b), the more evenly distributed velocity profile which has a label *b* makes the total kinetic energy a minimum. The most uneven distributions at points *a* and *c* correspond to the maximum kinetic energy. This energy oscillation comes from the relative discreteness of the chain with respect to the wavelength of the soliton. Until now, the continuum approximation and analytic wave form analysis have not shown this discreteness. Since the wiggle in energy occurs during the time it takes for the maximum compression to pass over two adjacent spheres, the frequency of the kinetic energy oscillation is related to the velocity of the soliton as $c=f\lambda$, where f is the frequency of the energy oscillation and λ is the effective

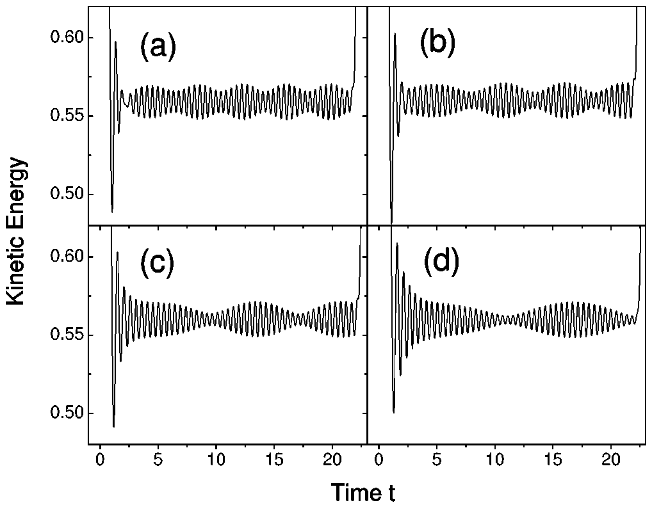

FIG. 7. Beats of kinetic energy when two solitons propagate along the chain with four different initial conditions: (a) $\Delta x_{12}(t=0)=0$, (b) $\Delta x_{12}(t=0)=0.1$, (c) $\Delta x_{12}(t=0)=0.2$, and (d) $\Delta x_{12}(t=0)=0.3$.

wavelength of the soliton in the energy transfer process. From Fig. 6, we find λ to be one grain diameter length which spans one or two grains.

This oscillation of kinetic energy also appears in the propagation of two solitons. As we mentioned previously, when two solitons propagate they have two different velocities, whether they had the same initial value or not. These different velocities will give rise to the oscillations of kinetic energy with different frequencies. Two neighboring frequency oscillations make beats as a function of the difference in frequencies. Figure 7 shows the beats which are generated during the propagation of two solitons.

IV. CONCLUSION

The multisoliton propagation problem in a linear Hertzian system shows various effects which were not seen in the propagation of a single soliton. After the boundary effect faded out, there exist some additional phenomena such as secondary soliton generation, the repulsion of two solitons, and a hierarchy in the velocity coefficient. We delineate these as coming from the interaction between the two solitons. And we find a kinetic energy oscillation, which comes from the discreteness of the granular chain. Two solitons of different velocities make beats. These oscillations and beats are generic to a granular chain.

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